Perturbations Produced in a Plasma by a Rapidly Moving Body

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The kinetic equation is used to find expressions for the Fourier components of the electron-density perturbation resulting from the passage of a body whose speed is much greater than the thermal velocity of the ions. The plasma is in a fixed magnetic field, such that the Larmor radius for the ions is much greater than the size of the body. The formulas are used to find the effective scattering cross section for electromagnetic waves whose wavelength is much greater than the size of the body.

1 General

The scattering of radio waves by a body moving in a plasma in a magnetic field has been given much attention recently in relation to artificial earth satellites.[1,2] We have considered this problem previously for zero magnetic field,[3] and such results as are necessary for this treatment are taken from that paper.

The plasma is assumed to be such that the mean free path l of the ions is much larger than the dimensions R_0 of the body and the wavelength λ:

$$l \gg R_0, \lambda \tag{1}$$

The total scattering is made up of the scattering from the body itself (e.g., a metallic sphere) and of that from the wake (the perturbed region) resulting from the passage of the body. The first component is given by the usual formulas of diffraction theory. The effects of collisions between particles are dealt with in Sec. 5. The perturbed region differs little in dielectric constant from the unperturbed region, so the metal sphere scatters much more than does a wake of the same size. Consequently the wake gives an effect comparable to that from the sphere only if it is much larger than the body; regions of size comparable to the wavelength play a major part in the scattering from the wake, whose structure changes only slowly over large distances.* The effects of still larger regions are much attenuated by interference between waves scattered from the various parts, so the wake can make a major contribution only if

$$\lambda \gg R_0 \tag{2}$$

Formula (7) will represent this condition more precisely. This is the only case considered.

The dielectric constant at large distances from the body is affected only slightly, and we may use perturbation theory, the applicability of which is discussed briefly in Sec. 6. For simplicity, we consider only the case in which the waves have a frequency ω much greater than the Larmor frequency of the electrons:

$$\omega \gg eH/mc \tag{3}$$

in which H is the magnetic field, e and m are the charge and mass of an electron, and c is the speed of light. The dielectric constant of the plasma may be taken as

$$\epsilon = 1 - (4\pi ne^2/m\omega^2)$$

and the change $\delta\epsilon$ is related to the change in electron density δn by

$$\delta\epsilon = -(4\pi e^2/m\omega^2)\delta n \tag{4}$$

Condition (3) is needed only to make (4) valid; the formulas given next for the Fourier components of δn still apply even if (3) is not obeyed. A standard formula in perturbation theory enables us to write directly the amplitude of the scattered wave at distances that are large relative to λ as

$$\mathbf{E}' = \frac{e^2}{m\omega^2\epsilon}\frac{e^{ikR}}{R}\mathbf{k}'[\mathbf{k}'\mathbf{E}_0]n_q \qquad |\mathbf{k}'| = k = \sqrt{\epsilon}(\omega/c) \tag{5}$$

Here E_0 is the amplitude of the incident wave, $\mathbf{k}'$ the wave vector of the scattered wave, ϵ the dielectric constant, and n_q a Fourier component of the electron-density perturbation:

$$n_q = \int \delta n(\mathbf{r}) \exp(-i\mathbf{qr})d^3r \qquad \mathbf{q} = \mathbf{k}' - \mathbf{k} \qquad |\mathbf{q}| = 2k\sin(\psi/2) \tag{6}$$

in which $\mathbf{k}$ is the wave vector of the incident wave and ψ the angle of scattering (the angle between $\mathbf{k}$ and $\mathbf{k}'$). Now λ appears in (6) only via $\mathbf{q}$, and (2) may be written more precisely as

$$qR_0 \ll 1 \tag{7}$$

The effective scattering cross section for an element of solid angle is

$$d\sigma = \frac{1}{16\pi^2\epsilon^2}\left(\frac{\omega_0}{\omega}\right)^4 \frac{|n_q|^2}{n_0^2} k^2 \sin^2\psi_1 do \qquad \omega_0^2 = 4\pi n_0^2 e^2/m \tag{8}$$

Here ψ' is the angle between k' and E_0, and n_0 the unperturbed electron density. For comparison, the effective cross section of metal sphere of radius R_0 is

$$d\sigma_m = (\omega/c)^4 R_0^6 \sin^2\psi_1 do \qquad \text{for} \qquad \lambda \gg 2\pi R_0 \tag{9}$$

The evaluation of (5) is then that of calculating the Fourier components. Gurevich[2] has discussed in some detail the perturbations a rapidly moving body produces in a plasma, and his results should, in principle, be applicable for n_q, but in practice it is simpler to derive n_q directly from the kinetic equation. Moreover, the solution is then more rigorous, for

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* The situation becomes more complicated if the body moves approximately in the direction of the magnetic field, for the rate of decay of the perturbed electron density varies with direction. In particular, the dimensions of the scattering region in the direction of the field may be much greater than the wavelength.

we can allow for certain neglected effects,[2] which are very important when a magnetic field is present.

Here we consider only the case in which the speed V_0 of the body lies between the thermal speeds of the electrons and ions:

$$\sqrt{\frac{kT}{M}} \ll V_0 \ll \sqrt{\frac{kT}{m}} \tag{10}$$

Then n is related to the electrical potential φ via Boltzmann's formula

$$n = n_0 \exp(e\varphi/kT) \tag{11}$$

The ion distribution must be found at the same time as the potential; we use the kinetic equation for the distribution of the ions in velocity and coordinates. Of course, it is of greater interest to solve the problem for finite magnetic field, but it is more convenient to start the deduction from the zero-field case. This is the subject of the next section, in which only simple physical arguments are used; a more rigorous derivation of the equations can be found elsewhere.[3]

2 Calculation of n_q without Allowance for the Magnetic Field

The function $f(\mathbf{u},\mathbf{r})$ for the distribution of the ions with respect to velocity and coordinates is

$$f(\mathbf{u},\mathbf{r}) = f'(\mathbf{u},\mathbf{r}) + f_0(\mathbf{u})$$
$$f_0(\mathbf{u}) = n_0 \left(\frac{M}{2\pi kT}\right) \exp\frac{-Mu^2}{2kT} \tag{12}$$

in which n_0 is the unperturbed density of the ions and M the mass of an ion. Distances much greater than R_0 are important in calculating n for a small q, but here f' and φ are small; the motion is described by a kinetic equation linearized with respect to f' and φ. The body and the region of strong field around it may be treated as a point, the presence of which is allowed for by inserting on the right-hand side a term, the meaning of which is an integral for the collisions between the ions and the body. This term is finite only at the position of the body; it takes the form

$$J(\mathbf{u})\delta(\mathbf{r})$$

in which $J(\mathbf{u})$ is some function of the ion speed. (We assume that the body always lies at the origin.) This implies that f' satisfies the following equation for a coordinate system as in the foregoing, in which f' is not a function of time [by virtue of (1) collisions between particles are neglected]:

$$\mathbf{v}\frac{\partial f'}{\partial \mathbf{r}} + \frac{e}{kT} f_0 \mathbf{u} \frac{\partial \varphi}{\partial \mathbf{r}} = J(\mathbf{u})\delta(\mathbf{r}) \tag{13}$$

Here $\mathbf{v} = \mathbf{u} - \mathbf{V}_0$ is the ion speed in the coordinate system in which the body is at rest. Furthermore we have taken into account the fact that

$$\frac{\partial f_0}{\partial \mathbf{u}} = -\frac{Mf_0}{kT}\mathbf{u}$$

The left-hand side in (13) is the total derivative of f with respect to time. If we recall the physical meaning of the distribution function, we see that $J(\mathbf{u})d^3u$ is the number of particles per unit time which, as a result of collisions with the body, acquire speeds in the range d^3u near $\mathbf{u}$. Then we merely repeat the argument used for the ordinary collision integral to get a formula for J, which is very useful in approximate calculations. We assume that the ions undergo elastic collisions with the body and are not neutralized. Let an ion passing near the body with a parameter ρ and an azimuthal angle φ acquire as a result a velocity $\mathbf{v}$; then the initial velocity is $\mathbf{v}_1(\mathbf{v},\rho,\varphi)$, in which the function is determined by the law of scattering, and the number of ions acquiring a velocity $\mathbf{v}$ in unit time is simply the number of ncide nt ions having the velocity $\mathbf{v}_1(\mathbf{v},\rho,\varphi)$:

$$\rho d\,\rho d\,\varphi v\, n_0 \left(\frac{M}{2\pi kT}\right)^{3/2} \exp\frac{-M[\mathbf{v}_1(\mathbf{v},\rho,\varphi) + \mathbf{V}_0]^2}{2kT} \tag{14}$$

The scattering is elastic, so that $|\mathbf{v}_1| = |\mathbf{v}|$, and ions at infinity have a Maxwellian distribution in a fixed coordinate system. To find $J(\mathbf{u})$ we subtract from (14) the number of ions of velocity $\mathbf{v}$ that are lost as a result of collision

$$\rho d\,\rho d\,\varphi v\, n_0 \left(\frac{M}{2\pi kT}\right)^{3/2} \exp\frac{-M(\mathbf{v} + \mathbf{V}_0)^2}{2kT} \tag{15}$$

Finally

$$J(\mathbf{u}) = J(\mathbf{v} + \mathbf{V}_0) = n_0 \left(\frac{M}{2\pi kT}\right)^{3/2} v \int \rho d\,\rho d\varphi \times \left\{\exp\frac{-M[\mathbf{v}_1(\mathbf{v},\rho,\varphi) + \mathbf{V}_0]^2}{2kT} - \exp\frac{-M(\mathbf{v} + \mathbf{V}_0)^2}{2kT}\right\} = f_0(\mathbf{u})v \int \rho d\,\rho d\varphi \left\{\exp\frac{-\mathbf{V}_0\Delta\mathbf{v}(\mathbf{v},\rho,\varphi)}{kT} - 1\right\} \tag{16}$$

in which $\Delta\mathbf{v} = \mathbf{v}_1 - \mathbf{v}$ is the change in velocity caused by the scattering.

That is, $J(\mathbf{u})$ can be found if we know the law of scattering as affected by the electric field of the body. Of course, (16) does not give us $J(\mathbf{u})$ in general, in part because the electric field around the body is not known. However, this difficulty can be avoided in certain of the more important cases. Consider, for example, a metal sphere whose radius R_0 is large relative to the Debye radius R_D. It has been shown[2] that the electric potential near the body is then of the order of $(kT/e)\ln(R_0/R_D)$, which is much less than the kinetic energy of the ions relative to the body $MV_0^2/2$:

$$e\varphi \ll MV_0^2/2$$

Thus we can neglect the electric field† with regard to $J(\mathbf{u})$. Moreover, the reflection coefficient of a metal sphere is fairly small for ions of moderate energy,[4] so one naturally assumes that ions striking the body are completely neutralized. Then only the second term remains in (16), and $J(\mathbf{u})$ is simply the number of ions striking the sphere in unit time:

$$J(\mathbf{u}) = \pi R_0^2|\mathbf{u} - \mathbf{V}_0|f_0 \approx \pi R_0^2 V_0 f_0(\mathbf{u}) \tag{17}$$

Of course, a body that acquires and neutralizes ions in this way must also acquire an equivalent number of electrons if its state is steady; otherwise its charge would change continuously. However, this uptake of electrons produces only a change of order $V_0(m/kT)^{1/2}$ in $n_\mathbf{q}$. Therefore the effect can be neglected.

The full system of equations related to (13) is found by adding Poisson's equation

$$\Delta\varphi = -4\pi e\left(\int f' d^3u - \delta n\right) \tag{18}$$

in which δn is related to φ by (11) in the linear approximation

$$\delta n = n_0(e\varphi/kT) \tag{19}$$

To (13, 18, and 19) we apply a Fourier transform—multiply by $\exp(-i\mathbf{q}\mathbf{r})$—and integrate with respect to d^3r:

$$i\mathbf{q}(\mathbf{u} - \mathbf{V}_p)f_\mathbf{q} + (e/kT)f_0 i\,\mathbf{q}\mathbf{u}\varphi_\mathbf{q} = J(\mathbf{u}) \tag{20}$$

$$q^2\varphi_\mathbf{q} = 4\pi e\left(\int{}^2 f_\mathbf{q} d^3u - n_\mathbf{q}\right) \tag{21}$$

$$n_\mathbf{q} = (n_0 e/kT)\varphi_\mathbf{q} \tag{22}$$

† This does not mean, of course, that we neglect the electric field generally, for it persists on the left-hand side in (13) and produces pronounced effects, especially in a magnetic field.

in which

$$f_{\mathbf{q}} = \int f' e^{-i\mathbf{q}\mathbf{r}}\, d^3r \qquad \varphi_{\mathbf{q}} = \int \varphi e^{-i\mathbf{q}\mathbf{r}} d^3r$$

We can neglect the term in q^2 in (21) if $q \ll 1/R_D$, so (21) and (22) are replaced by

$$n_{\mathbf{q}} = \int f_{\mathbf{q}} d^3u \qquad \varphi_{\mathbf{q}} = (kT/n_0e)n_{\mathbf{q}} \tag{23}$$

Then (20) and (23) give us $n_{\mathbf{q}}$. According to Landau[5] the integrals with $i\mathbf{q}(\mathbf{u} - \mathbf{V}_0)$ in the denominator are obtained by bypassing the singularity. This may be done by replacing $\mathbf{q}\mathbf{V}_0$ by $\mathbf{q}\mathbf{V}_0 + i\delta$, in which $\delta \rightarrow +0$; then

$$n_{\mathbf{q}} = \frac{1}{q}\left[\frac{1}{i}\int \frac{J(\mathbf{u})d^3u}{\mathbf{n}(\mathbf{u} - \mathbf{V}_0) - i\delta}\right]\left[1 + \left(\frac{M}{2\pi kT}\right)^{3/2} \times \int \frac{\mathbf{n}\mathbf{u}}{\mathbf{n}(\mathbf{u} - \mathbf{V}_0) - i\delta} \exp\frac{-Mu^2}{2kT}\, d^3u\right]^{-1} \quad (n = \mathbf{q}/q) \tag{24}$$

The integral in the denominator of (24) is best calculated in Cartesian coordinates that have the x axis along n. Now

$$\int_{\infty-}^{\infty} \frac{e^{-y^2}}{y - a - i\delta}\, dy = 2\sqrt{\pi}\left(i\frac{\sqrt{\pi}}{2} - \int_0^a e^{y^2}\, dy\right)e^{-a^2} \tag{25}$$

and finally

$$n_{\mathbf{q}} = \frac{1}{q}\left[\frac{1}{i}\int_0^a \frac{J(\mathbf{u})d^3u}{\mathbf{n}(\mathbf{u} - \mathbf{V}_0) - i\delta}\right]\left[2 - 2a\left(\int_0^a e^{x^2}\, dx - i\frac{\sqrt{\pi}}{2}\right)e^{-a^2}\right]^{-1} \left(a = \mathbf{n}\mathbf{V}_0\sqrt{M/2kT}\right) \tag{26}$$

This general formula has been given in Ref. 3. Substituting $J(\mathbf{u})$, as given by (17), in (26) we obtain for a metal sphere

$$n_{\mathbf{q}} = -\frac{\pi R_0^2 n_0}{q} \frac{\left(\frac{MV_0^2}{2kT}\right)^{1/2}\left(\frac{\sqrt{\pi}}{2} + i\int_0^a e^{x^2}\, dx\right)e^{-a^2}}{1 - ae^{-a^2}\int_0^a e^{x^2}\, dx + ia\frac{\sqrt{\pi}}{2}e^{-a^2}} \tag{27}$$

Also, (26) implies directly that the electron density decreases with $1/r^2$, which agrees with other results;[2] furthermore, the denominator in (26) varies rapidly with angle, for

$$e^{-a^2}\int_0^a e^{x^2}\, dx \approx \frac{1}{2a} \qquad \text{for} \qquad \mathbf{n}V_0 \sim V_0, a \gg 1$$

Here the denominator is unity. At those angles for which

$$a = \mathbf{n}\mathbf{V}_0\sqrt{(M/2kT)} \ll 1$$

the denominator is 2; this is an effect of the electric field.

The denominator would be 1 if the field were neglected; in that case we could obtain a formula valid for all q, not only for small values. In fact, f' satisfies a kinetic equation for the free motion of the ions whose right-hand side is supplemented by a term relating to the uptake of ions. Unit area absorbs in unit time the number of ions of velocity $\mathbf{v}$ that fall on it:

$$-f_0(\mathbf{v} + \mathbf{V}_0)(\mathbf{v}\mathbf{s}) \approx f_0(\mathbf{u})(\mathbf{s}\mathbf{V}_0)$$

in which $\mathbf{s}$ is the normal to the surface ($\mathbf{v}\mathbf{s} < 0$), and then f' satisfies

$$(\mathbf{u} - \mathbf{V}_0)\frac{\partial f}{\partial \mathbf{r}} = \begin{cases} -f_0\dfrac{\mathbf{V}_0\mathbf{r}}{r}\delta(r - R_0) & \text{for} \quad \mathbf{V}_0\mathbf{r} > 0 \\ 0 & \text{for} \quad \mathbf{V}_0\mathbf{r} < 0 \end{cases} \tag{28}$$

The Fourier transform of (28) is

$$i(\mathbf{u} - \mathbf{V}_0)\mathbf{q}f_{\mathbf{q}} = -f_0\pi R_0^2\cdot 2\int_0^{\pi/2} \sin\vartheta\, \cos\vartheta e^{iqR_0\cos\vartheta\cos\alpha} \times J_0(qR_0\sin\vartheta\,\sin\alpha)d\vartheta \tag{29}$$

in which α is the angle between V_0 and $\mathbf{q}$, J_0.

From (29):

$$n_{\mathbf{q}} = \frac{\pi R_0^2 n_0}{q}\left(\frac{MV_0^2}{2kT}\right)^{1/2}\left(\frac{\sqrt{\pi}}{2} + i\int_0^a e^{x^2}\, dx\right)e^{-a^2}\, 2 \times \int_0^{\pi/2} \sin\vartheta\, \cos\vartheta\; ep^{qR_0\cos\vartheta\cos\alpha}\, J_0(qR_0\sin\vartheta\,\sin\alpha)\, d\vartheta \tag{30}$$

This becomes (27) if $qR_0 \ll 1$, if we put the denominator as 1. We can put $\cos\alpha = 0$ in the integral on the right-hand side in (30) if α is close to $\pi/2$, whereupon

$$n_{\mathbf{q}} = \frac{\pi R_0^2 n_0}{q}\left(\frac{MV_0}{2kT}\right)^{1/2}\left(\frac{\sqrt{\pi}}{2} + i\int_0^a e^{x^2}\, dx\right)e^{-a^2} \times 2\frac{J_1(qR_0)}{qR_0}$$

Although we have assumed that $\cos\alpha \ll 1$, the parameter

$$a = \cos\alpha V_0\sqrt{M/2kT}$$

need not be small relative to 1.

3 Calculation of $n_{\mathbf{q}}$, Magnetic Field Present

The plasma is assumed to be in a constant magnetic field $\mathbf{H}$, whose strength is such that

$$R_0 \ll L_i = c\sqrt{MkT/eH} \tag{31}$$

in which L_i is the Larmor radius for the ions. The field does not affect the collisions between the body and the ions, so the right-hand side of (13) remains of the same form, and $J(\mathbf{u})$ is unchanged; on the left-hand side we must add a term for the Lorenz force on the ions. The initial equation is then

$$\frac{\partial f'}{\partial \mathbf{r}}(\mathbf{u} - \mathbf{V}_0) + \frac{\partial f}{\partial \mathbf{u}}\frac{e}{Mc}[\mathbf{u}\mathbf{H}] + \frac{e}{kT}f_0\,\mathbf{u}\frac{\partial \varphi}{\partial \mathbf{r}} = J(\mathbf{u})\delta(\mathbf{r}) \tag{32}$$

In Fourier components, we have instead of (20):

$$i\mathbf{q}(\mathbf{u} - \mathbf{V}_0)f_{\mathbf{q}} + \frac{e}{Mc}[\mathbf{u}\mathbf{H}]\frac{\partial f_{\mathbf{q}}}{\partial \mathbf{u}} + i\mathbf{q}\mathbf{u}\frac{e}{kT}f_0\varphi_{\mathbf{q}} = J(\mathbf{u}) \tag{33}$$

Here (23) retains its form. We transform (33) to cylindrical coordinates in velocity space, with the axis along the magnetic field, to obtain

$$\partial f_{\mathbf{q}}/\partial\beta = i(\alpha + \gamma\cos\beta)f_{\mathbf{q}} = B(\beta) \tag{34}$$

in which

$$\alpha = \frac{q_z u_z - \mathbf{q}\mathbf{V}_0}{\Omega} \qquad \gamma = \frac{q_1 u_1}{\Omega} \qquad \Omega = \frac{eH}{Mc}$$

$$B = \frac{ie}{\Omega kT}(\mathbf{q}\mathbf{u})f_0\varphi_{\mathbf{q}} - \frac{J}{\Omega}$$

where $\mathbf{q}_\perp$ and $\mathbf{u}_\perp$ are the projections of $\mathbf{q}$ and $\mathbf{u}$ on a plane normal to the magnetic field; β is the angle between these projections. A solution to (34) takes the form

$$f_{\mathbf{q}} = e^{i(\alpha\beta + \gamma\sin\beta)}\int_c^\beta B(t)e^{-i(\alpha t + \gamma\sin t)}\, dt$$

The constant c must be chosen to give a periodic function with respect to β. We put $c = \infty$, $t = x + \beta$ and obtain

$$f_{\mathbf{q}} = -\int_0^\infty \exp -i\{\alpha x + \gamma[\sin(\beta + x) - \sin\beta]\}B(x + \beta)dx$$

Trubnikov[6] has made similar calculations. We eliminate $f_{\mathbf{q}}$ and $\varphi_{\mathbf{q}}$ from (23) and (34) and alter the origin used for β (subscript $x/2$ indicates that the quantity must be taken with $\mathbf{q}_\perp$ turned through $x/2$):

$$n_{\mathbf{q}} = \frac{\frac{1}{\Omega}\int J(\mathbf{u})_{x/2} \exp - i\left\{\alpha x + 2\gamma\cos\beta \sin\frac{x}{2}\right\} dx\, d^3u}{1 + \frac{i}{\Omega}\int \frac{f_0}{n_0}(\mathbf{qu})_{x/2} \exp - i\left\{\alpha x + 2\gamma\cos\beta\sin\frac{x}{2}\right\} dx\, d^3u}$$

The integral in the denominator may be integrated by parts with respect to x; the result is

$$n_{\mathbf{q}} = \frac{\frac{1}{\Omega}\int J(\mathbf{u})_{x/2} \exp - i\left\{\alpha x + 2\gamma\cos\beta \sin\frac{x}{2}\right\} dx\, d^3u}{2 + i\frac{\mathbf{qV}_0}{\Omega}\int \frac{f_0}{n_0} \exp - i\left\{\alpha x + 2\gamma\cos\beta\sin\frac{x}{2}\right\} dx\, d^3u} \tag{35}$$

If

$$\Omega = \frac{eH}{Mc} \ll q_z u_z \sim q_z\sqrt{\frac{kT}{M}} \tag{36}$$

the integrand in (35) oscillates rapidly for $x \gtrsim 1$; small values of x are then important for this integral. We put $\sin x/2 \approx x/2$ and integrate with respect to x (here α is put as $\alpha + i\delta$, $\delta \to +0$), which gives us (24); thus (36) is the condition that allows us to neglect the magnetic field.

The integral with respect to d^3u in the denominator should be treated in cylindrical coordinates; the integral with respect to du_z is found directly, whereas the integral with respect to β represents a Bessel function of zero order. Then the integral with respect to du_1 is given by the formula

$$\int_0^\infty J_0(x)e^{-px^2}x\,dx = \frac{1}{2p}e^{-1/4p}$$

Finally

$$\int f_0 \exp - i\left[\frac{(q_z u_z - \mathbf{qV}_0)x + 2\mathbf{q}_\perp\mathbf{u}_\perp\sin(x/2)}{\Omega}\right]d^3u = n_0\exp\left[\frac{i\mathbf{qV}_0}{\Omega}x - \frac{kT}{2M\Omega^2}\left(q_z^2x^2 + 4q_\perp^2\sin^2\frac{x}{2}\right)\right] \tag{37}$$

and

$$n_{\mathbf{q}} = \frac{\frac{1}{\Omega}\int J(\mathbf{u})_{x/2}\exp - i\left[\alpha x + 2\gamma\cos\beta\sin\frac{x}{2}\right]dx\,d^3u}{2 + \frac{i}{\Omega}\mathbf{qV}_0\int_0^\infty \exp\left[i\frac{\mathbf{qV}_0}{\Omega}x - \frac{kT}{2M\Omega^2}\left(q_z^2x^2 + 4q_\perp^2\sin^2\frac{x}{2}\right)\right]dx} \tag{38}$$

Formula (38) corresponds to (26), which relates to zero field.

We substitute (17) into (38) and calculate the integral in the numerator by means of (37):

$$n_{\mathbf{q}} = -\pi R_0^2 n_0 V_0\left[\frac{1}{\Omega}\int_0^\infty \exp\left\{i\frac{\mathbf{qV}_0}{\Omega}x - \frac{kT}{2M\Omega^2}\times\left(q_z^2 + 4q_\perp^2\sin^2\frac{x}{2}\right)\right\}dx\right]\times\left[2 + i\frac{\mathbf{qV}_0}{\Omega}\int_0^\infty\times\exp\left\{i\frac{\mathbf{qV}_0}{\Omega}x - \frac{kT}{2M\Omega^2}\left(q_z^2x^2 + 4q_\perp^2\sin^2\frac{x}{2}\right)\right\}dx\right]^{-1} \tag{39}$$

This expression, as we shall see, has a sharp peak when $q_z \to 0$, $\mathbf{qV}_0 \to 0$, so we shall examine the behavior of $n_{\mathbf{q}}$ near these values of $\mathbf{q}$ in some detail. Let

$$\frac{q_z}{\Omega}\sqrt{\frac{kT}{M}} \ll 1 \quad \text{and} \quad \frac{\mathbf{qV}_0}{\Omega} \ll 1 \tag{40}$$

Subject to (40), the function

$$\exp\frac{-2q_\perp^2kT}{M\Omega^2}\sin^2\frac{x}{2}$$

oscillates much more rapidly than the other factors in the integrand in (39); thus we can replace this function by its mean taken over a period, as in

$$\exp\frac{-2q_\perp^2kT\sin^2\frac{x}{2}}{M\Omega^2} \to \frac{1}{2\pi}\int_0^{2\pi}\exp\frac{-q_\perp^2kT}{M\Omega^2}\times(1-\cos x)dx = \exp\left(-\frac{q_1^2kT}{M\Omega^2}\right)I_0\left(\frac{q_1^0kT}{M\Omega^2}\right) \tag{41}$$

in which I_0 is a Bessel function of imaginary argument. The integrals in (39) can now be transformed by means of

$$\int_0^\infty e^{ita-(t^2/4)}dt = 2\left(\frac{\sqrt{\pi}}{2} + i\int_0^a e^{x^2}dx\right)e^{-a^2} \tag{42}$$

to give

$$n_{\mathbf{q}} = -\frac{\pi R_0^2V_0n_0}{q_z}\left[\sqrt{\frac{M}{2kT}}\left(\sqrt{\pi} + 2i\int_0^{a_1}e^{x^2}dx\right)\times e^{-a_1^2}\exp\frac{-q_\perp^2kT}{M\Omega^2}I_0\left(\frac{q_\perp^2kT}{M\Omega^2}\right)\right]\times\left[2 + ia_1\times\left(\sqrt{\pi} + 2i\int^{a_1}e^{x^2}dx\right)e^{-a_1^2}\exp\frac{-q_\perp^2kT}{M\Omega^2}I_0\left(\frac{q_\perp^2kT}{M\Omega^2}\right)\right]^{-1} \tag{43}$$

in which

$$a_1 = \frac{\mathbf{qV}_0}{q_z}\sqrt{\frac{M}{2kT}} = \left(V_{z0} + \frac{\mathbf{q}_\perp\mathbf{V}_{0\perp}}{q_z}\right)\sqrt{\frac{M}{2kT}} \tag{44}$$

Eq. (43) shows that $n_{\mathbf{q}}$ goes to infinity as $1/q_z$ if $q_z \to 0$ and $|\mathbf{q}_\perp\mathbf{V}_\perp|/q_z < \infty$. Here the special case is that of a body moving precisely along the magnetic field; then $\mathbf{q}_\perp\mathbf{V}_\perp = 0$ and a_1 is not dependent on $\mathbf{q}$:

$$a_1 = V_0\sqrt{\frac{M}{2kT}}$$

Now

$$V_0\sqrt{\frac{M}{2kT}} \ll 1$$

Thus (44) can be written

$$n_{\mathbf{q}} = -\frac{\pi R_0^2n_0}{q_z}\left[i\exp\left(-\frac{q_\perp^2kT}{M\Omega^2}\right)I_0\left(\frac{q_\perp^2kT}{M\Omega^2}\right)\right]\times\left[2 - \exp\left(\frac{-q_\perp^2kT}{M\Omega^2}\right)I_0\left(\frac{q_\perp^2kT}{M\Omega^2}\right)\right]^{-1} \tag{45}$$

That is, $n_{\mathbf{q}} \to \infty$ as $q_z \to 0$ no matter what $\mathbf{q}_\perp$ may be if the body moves along the field. The simple physical meaning of this is that the perturbed region behind the body cannot revert to its former state if there are no collisions, due to the action of the field; the track takes the form of a cylinder the axis of which lies along $\mathbf{H}$; thus $n_{\mathbf{q}}$ increases if $\mathbf{q} \perp \mathbf{H}$.

Now if $\mathbf{q}_\perp \mathbf{V}_\perp \neq 0$, we have for $q_z = 0$:

$$n_\mathbf{q} = \frac{\pi R_0^2 V_0}{\mathbf{q}_\perp \mathbf{V}_{\perp 0}} \left[i \exp\left(-\frac{q_\perp^2 kT}{M\Omega^2} \right) I_0\left(\frac{q_\perp^2 kT}{M\Omega^2} \right) \right] \times \left[2 - \exp\left(-\frac{q_\perp^2 kT}{M\Omega^2} \right) I_0\left(\frac{q_\perp^2 kT}{M\Omega^2} \right) \right]^{-1} \quad (46)$$

The numerator in (39) becomes infinite when $q_z \to 0$, $\mathbf{q}\mathbf{V}_0 \to p\Omega$, and $|\mathbf{q}\mathbf{V}_0 - p\Omega|/|q_z| < \infty$ (p is an integer), but the denominator then becomes infinite too, the result being that $n_\mathbf{q}$ remains finite and approaches $i\pi R_0^2 V_0 n_0/p\Omega$. The result is finite due to the electric field, for the denominator would be precisely 1 if this field were neglected. Therefore the field is very important in this case.

The formula applicable for any q_z and $\mathbf{q}\mathbf{V}_0$ provided that $q_\perp^2 kT/M\Omega^2 \ll 1$ is

$$n_\mathbf{q} = -\frac{\pi R_0^2 n_0 V_0}{q_z} \times \frac{\sqrt{\dfrac{M}{2kT}} \left(\sqrt{\pi} + 2i \displaystyle\int_0^{a_1} e^{x^2}\, dx \right) e^{-a_1^2}}{2 + i a_1 \left(\sqrt{\pi} + 2i \displaystyle\int_0^{a_1} e^{x^2}\, dx \right) e^{-a_1^2}} \quad (47)$$

4 Calculation of $n_\mathbf{q}$ for a Point Charge

Consider a body whose size is much less than R_0 [(53) gives the condition more precisely]; Kraus and Watson[7] have considered this case, which is of interest mainly as regards the method. We shall see that their formulas[7] are incorrect for $\mathbf{q}$ small (that is, for large distances). First we find $J(\mathbf{u})$ by means of (16). The main contribution to the integral with respect to ρ comes from values of $\rho \gtrsim R_D$ if the charge is small; the field at such distances is purely of coulomb type. The angle ϑ between $\mathbf{v}$ and $\mathbf{v}_1$ is small if the charge Q is small; thus

$$\vartheta = \frac{2Qe}{Mv^2} \frac{1}{\rho} \approx \frac{2Qe}{MV_0^2 \rho} \quad (48)$$

We expand the right-hand side of (16) as a series in ϑ as far as terms of order ϑ^2; this is possible if the exponents containing ϑ in (16) are small, and so we have

$$\sqrt{\frac{MV_0^2}{kT}}\, \vartheta \sim \frac{Qe}{\sqrt{MV_0 kT}} \frac{1}{\rho} \ll 1 \quad (49)$$

This condition must, in any case, be complied with for $\rho \gtrsim 1/R_D$; thus the condition on the charge is

$$\frac{Qe}{\sqrt{MV_0^2 kT}} \frac{1}{R_D} \ll 1 \quad (50)$$

Expanding and integrating with respect to φ, we have

$$J = 2\pi f_0 V_0 \times \left\{ \frac{MV_0^2}{2kT} + \frac{M}{4(kT)^2} [\mathbf{V}_0^2 \mathbf{v}^2 - (V_0 \mathbf{v})^2] \right\} \int \vartheta^2(\rho) \rho\, d\rho \quad (51)$$

The integral on the right-hand side in (51) diverges logarithmically; it should be terminated at ρ of the order of R_D for large ρ and at the value of ρ for which (49) ceases to be obeyed for small ρ; that is, at

$$\rho \sim \rho_1 = Qe/\sqrt{MV_0^2 kT}$$

and so

$$J(\mathbf{u}) = -\left\{ \frac{4\pi Q^2 e^2}{MkT} f_0 \ln \frac{R_D}{\rho_1} \frac{1}{V_0^3} \left\{ V_0^2 - \frac{M}{2kT} [V_0^2 u^2 - (\mathbf{V}_0 \mathbf{u})^2] \right\} \right. \quad (52)$$

It is assumed that the field follows the inverse-square law down to ρ of the order of ρ_1, so the size of the body must satisfy

$$R_0 \gtrsim \rho_1 = \frac{Qe}{\sqrt{MV_0^2 kT}} \quad (53)$$

We substitute (52) into (26) and (38) to obtain for zero field and finite magnetic field, respectively:

$$n_\mathbf{q} = -\frac{1}{q} \frac{2\pi Q^2 e^2 n_0}{V_0^3 (2k^3 T^3 M)^{1/2}} \ln\frac{R_D}{\rho_1} [V_0^2 - (\mathbf{V}_0 \mathbf{u})^2] \times \left[(1 - 2a^2)\left(\sqrt{\pi} + 2i \int_0^a e^{x^2}\, dx \right) e^{-a^2} + 2ia \right] \times \left[2 - 2a\left(\int_0^a x^2 dx - i\frac{\sqrt{\pi}}{2} \right) e^{-a^2} \right]^{-1} \quad (54)$$

$$n_\mathbf{q} = \frac{2\pi Q^2 e^2 n_0}{MkTV_0^3} \ln\frac{R_D}{\rho_1} \left[\frac{kT}{M\Omega^2} \int_0^\infty V_0^2 \left(q_z^2 x^2 + q_\perp^2\, 4\sin^2\frac{x}{2} \right) - \left(q_z V_{0z} x + q_\perp V_{1\perp}\, 2\sin\frac{x^2}{2} \right) \times \exp\{ \dots \} dx \right] \left[2 + i\frac{\mathbf{q}\mathbf{V}_0}{\Omega} \int_0^\infty \exp\left\{ i\frac{qV_0}{\Omega} - \frac{kT}{2M\Omega^2} \left(q_z^2 x^2 + 4q_\perp^2 \sin^2\frac{x}{2} \right) \right\} dx \right]^{-1} \quad (55)$$

Here $\mathbf{V}_{1\perp}$ is the vector derived from $\mathbf{V}_{0\perp}$ by rotation through $-x/2$ (the braces in the numerator of (55) denote the expression in braces in the denominator).

We see that (54) and (55) contain Q^2, so they become zero in the first approximation for perturbations with respect to Q, as in Kraus and Watson's treatment. This means that this treatment gives the wrong result for small q, that is, for large distances, to which (54) applies. To (54) and (55) we should add (for large q) expressions derived from the foregoing first approximation. The formulas for zero field and for finite magnetic field are, respectively

$$n_\mathbf{q} = \frac{Q/e}{2(R_D q)^2 + 2 - 2a\left(\displaystyle\int_0^a e^{x^2}\, dx - \frac{i\sqrt{\pi}}{2} \right) e^{-a^2}} \quad (56)$$

and

$$n_\mathbf{q} = \frac{Q/e}{2(R_D q)^2 + 2 + i\dfrac{\mathbf{q}\mathbf{V}_0}{\Omega} \displaystyle\int_0^\infty \exp\left\{ i\frac{\mathbf{q}\mathbf{V}_0}{\Omega} x - \frac{kT}{2M\Omega^2} \left(q_z^2 x^2 + 4q_\perp^2 \sin^2\frac{x}{2} \right) \right\} dx} \quad (57)$$

We shall not determine here the ranges of $\mathbf{q}$ in which (54) and (55) give the main contribution or those for (56) and (57).

5 Correction for Finite Mean Free Path

Here we estimate the effect of collisions between particles on $n_\mathbf{q}$; to do this we insert the integral for the collisions of ions with particles of any kind into (20) and (22). The exact form of this integral is very complicated, so we restrict the treatment to the insertion of the effective collisional frequency ν. The usual collision integral is expressed in terms of ν as $Y = -\nu(f - f_0)$, but this form is unsatisfactory here, for it conflicts with the constancy of the number of particles ($\int Y d^3u \neq 0$), whereas we are concerned with the distribution of the particles in space. Thus we put the collision integral in the form

$$Y = -\nu\left(f - \frac{f_0}{n_0}\int f d^3u\right) = -\nu\left(f' - \frac{f_0}{n_0}\int f' d^3u\right) \quad (58)$$

and then

$$\int Y d^3u \equiv 0$$

The integral of (58) becomes zero for the Boltzmann distribution, as it should, since

$$f = e^{-(e\varphi/kT)} f_0$$

Now ν is related to the diffusion coefficient D (for zero field) by

$$D = 2kT/\nu M \quad (59)$$

We add the expression in (58) to the right-hand side of (20) and, solving for $n_{\mathbf{q}}$ for zero magnetic field, we have

$$n_{\mathbf{q}} = \left[\frac{1}{i}\int \frac{J(\mathbf{u})d^3u}{\mathbf{q}(\mathbf{u} - \mathbf{V}_0) - i\nu}\right]\left[2 + \frac{(iqV_0 - 2\nu)}{n_0} \times \int \frac{f_0 d^3u}{i\mathbf{q}(\mathbf{u} - \mathbf{V}_0) + \nu}\right]^{-1} \quad (60)$$

From (60) we see that the collisions are negligible if one of the following conditions is obeyed:

$$\mathbf{q}V_0 \gg \nu \qquad q\sqrt{\frac{kT}{M}} \gg \nu \quad (61)$$

In the converse limiting case

$$q\sqrt{\frac{kT}{M}} \ll \nu \qquad \mathbf{q}\mathbf{V}_0 \ll \nu$$

which is, perhaps, only of interest for the method

$$n_{\mathbf{q}} = \frac{1}{2}\left[\int J(\mathbf{u})d^3u - i\frac{q}{\nu}\int \mathbf{u}J(\mathbf{u})d^3u\right] \times \left[-i\mathbf{q}\mathbf{V}_0 + \frac{q^2kT}{\nu M}\right]^{-1} \quad (62)$$

The second term in the numerator has been retained as being the main one when $\int J(\mathbf{u})d^3u = 0$.

Now (37) shows that the numerator in the expression for $n_{\mathbf{q}}$ remains finite as $q \to 0$, whereas the denominator becomes infinite; this is a result of the slower decay of the perturbation in the region to which the diffusion equation applies.‡

We substitute the $J(\mathbf{u})$ of (17) into (60) to replace (26) by

$$n_{\mathbf{q}} = -\frac{\pi R_0^2 n_0}{q} \times \frac{\left(\frac{MV_0^2}{2kT}\right)^{1/2}\left(\frac{\sqrt{\pi}}{2} + i\int_0^{a_2} e^{x^2}dx\right)e^{-a_2^2}}{2 + (a_2 + i\nu)\sqrt{M/2kTq^2}\left(\frac{\sqrt{\pi}}{2} + i\int_0^{a_2} e^{x^2}dx\right)e^{-a_2^2}} \quad (63)$$

$$a_2 = \left(\mathbf{n}\mathbf{V}_0 + \frac{i\nu}{q}\right)\sqrt{\frac{M}{2kT}}$$

Then (57) is added to the right-hand side of (32), and the equation is solved with $J(\mathbf{u})$ in the form of (17), to obtain in place of (39):

$$n_{\mathbf{q}} = -\frac{\pi R_0^2 n_0 V_0}{\Omega}\left[\int_0^\infty \exp\left\{\frac{i\mathbf{q}\mathbf{V}_0 - \nu}{\Omega}x - \frac{kT}{2M\Omega^2} \times \left(q_z^2x^2 + 4q_\perp^2\sin^2\frac{x}{2}\right)\right\}dx\right] \times \left[2 + \frac{(i\mathbf{q}\mathbf{V}_0 - 2\nu)}{\Omega} \times \int_0^\infty \exp\left\{\frac{i\mathbf{q}\mathbf{V}_0 - \nu}{\Omega}x - \frac{kT}{2M\Omega^2} \times \left(q_z^2x^2 + 4q_\perp^2\sin^2\frac{x}{2}\right)\right\}dx\right]^{-1} \quad (64)$$

In the case $q_z = 0$ we have

$$n_{\mathbf{q}} = -\frac{\pi R_0^2 n_0 V_0}{2\nu}\left[\exp\left(-\frac{q_\perp^2kT}{M\Omega^2}\right)I_0\left(\frac{q_\perp^2kT}{M\Omega^2}\right)\right] \times \left[1 - \exp\left(-\frac{q_\perp^2kT}{M\Omega^2}\right)I_0\left(\frac{q_\perp^2kT}{M\Omega^2}\right)\right]^{-1} \quad (65)$$

Here we must remember that the results of this section have not been derived rigorously, since the collision integral for ions in a plasma does not, in fact, have the form of (57). Further study of the correction for collision between particles, of the applicability of the results to a body whose size is not small relative to R_D, and of the effects of the electric field on $J(\mathbf{u})$ is needed.

6 Effective Scattering Cross Section for Electromagnetic Waves

The foregoing formulas for $n_{\mathbf{q}}$ may be used with (3) to find the effective scattering cross section of the wake. Here (27) (no magnetic field) gives

$$d\sigma = \frac{1}{16}\left(\frac{\omega_0}{c}\right)^4\frac{R_0^4\sin^2\psi}{q^2} \times \frac{b^2\left[\frac{\pi}{4}e^{-2a^2} + \left(e^{-a^2}\int_0^a e^{x^2}dx\right)^2\right]}{\left(1 - ae^{-a^2}\int_0^a e^{x^2}dx\right)^2 + \frac{\pi}{4}e^{-2a^2}}do = \frac{1}{16}\left(\frac{\omega_0}{c}\right)^4\frac{R_0^4\sin^2\psi}{q^2}\Phi(b,\alpha)do \quad (66)$$

in which α is the angle between $\mathbf{q}$ and $\mathbf{V}_0$, $a = b\cos\alpha$, and $b = (MV_0^2/2kT)^{1/2}$. Figure 1 shows $\Phi(\alpha)$ for several b in polar coordinates, in which α is read clockwise from the vertical axis; the radius vector represents $\Phi(\alpha)$. Formula (66) (for a given ψ) has a sharp peak near $\alpha = \pi/2$, or, more precisely, at $|\alpha - \pi/2| \lesssim (2kT/MV_0^2)^{1/2}$. If α is not close to $\pi/2$ (and ψ is not zero), $d\sigma$ is given roughly by

$$d\sigma \simeq \left(\frac{\omega_0}{c}\right)^4\frac{R_0^4c^2}{\epsilon\omega^2}do \quad (67)$$

The scattering cross section for $\alpha \approx \pi/2$ is

$$d\sigma \simeq \frac{MV_0^2}{kT}\frac{R_0^4c^2}{\epsilon\omega^2}do \quad (68)$$

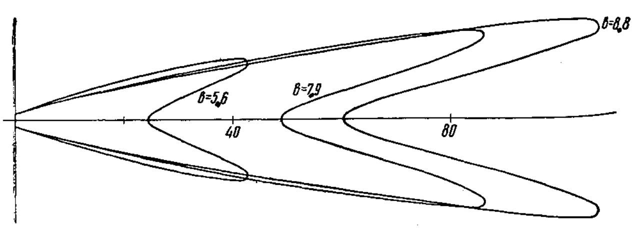

Fig. 1

‡ A. V. Gurevich pointed out this effect.

which is much larger. The case $\alpha = \pi/2$ corresponds to reflection from the direction of $\mathbf{V}_0$; the reason for this is that the wake is very much elongated along $\mathbf{V}_0$. There is also a sharp rise in $d\sigma$ as $q \rightarrow 0$, that is, as $\psi \rightarrow 0$.

Now we consider briefly the applicability of (8) (from perturbation theory). This formula is applicable only if $E' \ll E_0$ for large distances. It can be shown that, in our case, distances R of the order of $1/q$ for which $E' \approx (\omega_0/c)^2 R_0^2 E_0$ are important. The condition is

$$\left(\frac{\omega_0}{c} R_0\right)^2 \ll 1$$

with which it is almost always possible to comply when (7) is complied with ($\omega > \omega_0$ on all occasions).

The formula for finite magnetic field is found directly by incorporating the finite mean free path of the ions. We insert (64) into (8) in conjunction with the symbols

$$\beta = \frac{q\mathbf{V}_0}{\Omega} \qquad \mu = \frac{\nu}{\Omega}$$

$$F_1 = \int_0^\infty \cos\beta x \exp\left\{-\mu x - \frac{kT}{2M\Omega^2} \times \left(q_z^2 x^2 + 4q_\perp^2 \sin^2\frac{x}{2}\right)\right\} dx \quad (69)$$

$$F_2 = \int_0^\infty \sin\beta x \exp\left\{-\mu x - \frac{kT}{2M\Omega^2} \times \left(q_z^2 x^2 + 4q_\perp^2 \sin^2\frac{x}{2}\right)\right\} dx$$

to obtain

$$d\sigma = \frac{1}{16}\left(\frac{\omega_0}{c}\right)^4 \frac{R_0^4 V_0^2 \sin^2\psi}{\Omega^2} \times \frac{F_1^2 + F_2^2}{(2 - 2\mu F_1 - \beta F_2)^2 + (\beta F_1 - 2\mu F_2)^2} do \quad (70)$$

Numerical methods are required to obtain detailed values of $d\sigma$ from (70). The results of numerical integration form the subject of a later paper.

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Reviewer's Comment

This paper is an extension of previous work[1] on the scattering of radio waves by a body moving in an isotropic plasma. The main purpose of the analysis is to determine the contributions to scattering of an incident em wave due to the wake of a moving body in a plasma such as a satellite in the ionosphere. In order to deduce a scattering cross section for the wake, it is necessary to determine the perturbations in electron distribution caused by the moving body. Herein lies the main contribution of this article, namely, in the use of the kinetic equation and physical arguments to arrive at an approximate (semitractable) expression for the perturbation in electron density. Further considerations are given to the effect of magnetic fields and collisions on the nature of the perturbed trail. Finally, expressions for the scattering cross section both with and without a magnetic field are deduced. The plot of the variation of the pertinent functional parameters used in calculating the scattering cross section (in the case of zero magnetic field) show an interesting variation with vehicle velocity and ion temperature in that the maxima appear at angles slightly off the direction of specular reflection (mirror image reflection).

The work complements the contributions in the American scientific literature (for a substantial list of references see the review articles by Chopra[2] and Zachary[3]) on potential distributions and perturbations caused by moving vehicles in plasma. It is one of the very few attempts to relate the induced perturbations to electromagnetic effects which may be observable by an Earth-based radio system.

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